J. Fluid Mech. (1967), *vol.* 28, *part* 4, *pp.* 755–767
Printed in Great Britain

Surface-wave scattering matrix for a shelf

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(Received 31 May 1966 and in revised form 22 August 1966)

The diffraction of gravity waves at a discontinuity in depth is described by a scattering matrix that relates the asymptotic, plane-wave fields (each of which may contain waves travelling towards and away from the discontinuity) on the two sides of the discontinuity. Plane-wave and variational approximations for the elements of this scattering matrix are developed. These approximate results are tested by comparison with the more accurate results obtained by Newman for an infinite step. The plane-wave approximation to the magnitude of the transmission coefficient is within 5 % of Newman's result for all wavelengths, but the corresponding approximation to the reflexion coefficient is satisfactory only for rather long wavelengths. The variational approximations to the complex transmission and reflexion coefficients agree with Newman's results, within the accuracy with which his graphs can be read, for all wavelengths. The variational approximations also are used to determine the effects of trapped modes on the resonant width of a shelf that terminates at a vertical cliff.

1. Introduction

We consider the diffraction of gravity waves at a discontinuous change in depth (vertical step) between two horizontal bottoms and obtain some new results for obliquely incident waves and for a shelf of finite width. Our primary purpose, however, is to develop and illustrate a scattering-matrix formulation and an associated variational principle, due originally to Schwinger (1944), that have proved powerful and efficient in the treatment of acoustical (Miles 1946) and electromagnetic (Marcuvitz 1951) scattering problems (the formulations of these scattering problems are in terms of equivalent circuits, which, in turn, can be represented by scattering matrices). The application of Schwinger's variational technique to gravity-wave problems has been considered previously by Keller (1952), but his work has never been published (Prof. Keller informs me that the original manuscript has been lost).

The shallow-water problem for a step was treated originally by Lamb (1932, § 176), who invoked basic continuity requirements to relate the disturbances at large distances (compared with the depth) from the step. Bartholomeusz (1958) gave a more complete analysis and formulated the integral equation that governs the problem for arbitrary depths, but he solved this integral equation only in the

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limiting case of long waves to obtain reflexion and transmission coefficients that are identical with those of Lamb.† Both Lamb and Bartholomeusz restricted their analyses to normally incident waves. Sretenskii (1950) has considered waves obliquely incident on an infinite step between deep and shallow bottoms, but his analysis is inconsistent (Newman 1965*a*, *b*). Newman has considered waves normally incident on an infinite step and given numerical results for the reflexion and transmission coefficients over the complete range of wavelengths in which diffraction is significant (diffraction obviously cannot be significant if the depth is large compared with the wavelength on both sides of the step). Newman's numerical results, which were based on the solution of as many as eighty simultaneous equations on a digital computer, provide an excellent standard of comparison for our more approximate variational calculations.

The extension of the variational formulation to other configurations (e.g. scattering from obstacles) can be carried out more or less by analogy with comparable wave-guide problems (Marcuvitz 1951). See, for example, the calculation of the resonant frequencies of a harbour that is coupled to the open sea through an aperture in a breakwater (Miles & Munk 1961).

Being concerned primarily with illustrating the variational formulation, we do not offer comparisons with either laboratory or field observations. The comparisons given by Newman suggest that the idealized model on which our calculations are based is capable of providing results that are in fair agreement with observation.

2. The boundary-value problem

We consider small-amplitude, monochromatic, irrotational motion of an ideal liquid with an equilibrium free surface at $y = 0$ over the stepped bottom sketched in figure 1 ($h_2 > h_1$ with no loss of generality). We follow Newman (1965*a*) in our initial formulation and notation except as noted; in particular, we begin by regarding the motion as two-dimensional, although we subsequently generalize to three-dimensional motion. We pose the time dependence $e^{-i\sigma t}$ and the complex velocity potential ϕ, such that the particle velocity is given by

$$\mathbf{V} = \mathrm{Re}\,[e^{-i\sigma t}\nabla\phi(x,y)], \quad \nabla^2\phi = 0. \tag{2.1 a, b}$$

The (linearized) free-surface and bottom conditions are

$$\partial\phi/\partial y + K\phi = 0 \quad (K = \sigma^2/g,\ y = 0), \tag{2.2}$$

and $$\partial\phi_m/\partial y = 0 \quad (y = h_m), \qquad \partial\phi_2/\partial x = 0 \quad (x = 0,\ h_1 < y < h_2), \tag{2.3 a, b}$$

where $m = 1(2)$ denotes the solution for $x < (>)\,0$; (2.3*a*) must be replaced by an appropriate finiteness condition in the limit of infinite depth for $x > 0$ ($h_2 \to \infty$).‡

† Bartholomeusz, pointing to Lamb's disregard of the boundary condition on the vertical portion of the step, expressed surprise at this agreement. In fact, the explanation given by Lamb (in a footnote) appears to be perfectly sound; cf. Rayleigh's (1945) more extensive discussion in connexion with similar approximations in acoustical diffraction problems.

‡ The boundary-value problem for the total domain is defined by (2.1)–(2.3). The subsequent subdivision of the problem for the two domains, $x > 0$ and $x < 0$, requires the addition of the matching conditions across $x = 0$, (2.11) below.

A particular solution to (2.1), (2.2) and (2.3a) for a domain of constant depth is given by

$$\phi(x,y;k) = e^{kx}\psi(y,k), \tag{2.4}$$

$$\psi(y,k) = 2^{\frac{1}{2}}[h-K^{-1}\sin^2 kh]^{-\frac{1}{2}}\cos[k(h-y)], \tag{2.5}$$

and

$$k\tan kh + K = 0. \tag{2.6}$$

The eigenvalue equation (2.6) has a single pair of imaginary roots, say $\pm k_0 = \pm i\kappa$ ($\kappa > 0$), and an infinite, discrete set of real roots, say $\pm k_s$, $s = 1, 2, \ldots$; the corresponding functions of (2.5) form a complete, orthonormal set for the interval $y = (0, h)$ with the end conditions (2.2) and (2.3a). We append the subscript $m = 1, 2$ to h, ϕ and ψ in the subsequent discussion with the implication that the

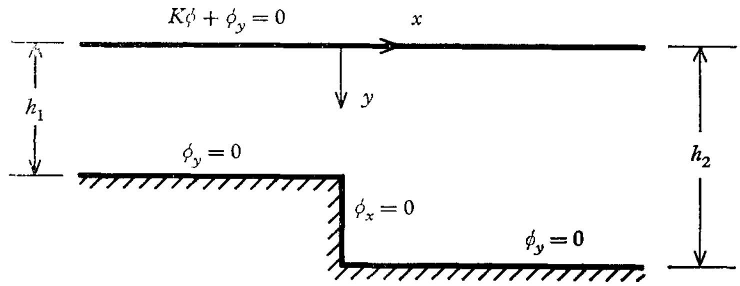

FIGURE 1. The boundary-value problem for a stepped bottom under a free surface; ϕ is a harmonic function and $K = \sigma^2/g$, where σ is the angular frequency of the gravity waves on the surface.

corresponding spectrum of k is defined by setting $h = h_m$ in (2.6); however, in order to avoid unwieldy subscripts, we omit the m subscript from the real eigenvalues. We find it expedient to introduce special notation for the surface-wave modes, such that

$$\chi_m(y) \equiv \psi_m(y, k_0) = 2^{\frac{1}{2}}[h_m + K^{-1}\sinh^2\kappa_m h_m]^{-\frac{1}{2}}\cosh[\kappa_m(h_m - y)] \tag{2.7}$$

and

$$\kappa_m \tanh \kappa_m h_m = K. \tag{2.8}$$

We note that $\kappa > K$ for finite h ($\kappa_1 = K_0$ in Newman's formulation). The spectrum for $h_2 = \infty$ goes over to $\kappa_2 = K$ and $0 < k < \infty$, with†

$$\chi_2(y) = (2K)^{\frac{1}{2}}e^{-Ky}, \quad \psi_2(y,k) = (2/\pi)^{\frac{1}{2}}(k^2+K^2)^{-\frac{1}{2}}(K\sin ky - k\cos ky) \quad (h_2 = \infty). \tag{2.9a,b}$$

We construct complete solutions for the individual domains in the form

$$\phi_m(x,y) = \operatorname{sgn} x[(A_m e^{-i\kappa_m|x|} + B_m e^{i\kappa_m|x|})\chi_m(y) + \sum_k C_m(k)e^{-k|x|}\psi_m(y,k)], \tag{2.10}$$

where the summations are over the positive-real eigenvalues (if $h_2 = \infty$ the summation for ϕ_2 is replaced by integration from $k = 0$ to $k = \infty$). We must determine A_m, B_m and $C_m(k)$ subject to the matching conditions

$$\partial\phi_1/\partial x = \partial\phi_2/\partial x, \quad \phi_1 = \phi_2 \quad (x = 0,\ 0 < y < h_1), \tag{2.11a,b}$$

† We remark that $\psi_2(y, k)$ does *not* satisfy a radiation condition as $y \to \infty$.

together with (2.3b). Let $U(y)$ denote the horizontal component of $\mathbf{V}$ in the plane $x = 0$; equating both $\partial\phi_1/\partial x$ and $\partial\phi_2/\partial x$, as given by (2.10), to $U(y)$ at $x = 0$ and invoking (2.3b) and the orthogonality of the $\psi_m(y,k)$, we obtain the Fourier coefficients

$$-i\kappa_m(A_m - B_m) = \int_0^{h_1} U(\eta)\chi_m(\eta)\,d\eta \tag{2.12}$$

and

$$-kC_m(k) = \int_0^{h_1} U(\eta)\psi_m(\eta,k)\,d\eta. \tag{2.13}$$

Substituting (2.13) into (2.10) and invoking (2.11b), we obtain the integral equation

$$(A_1+B_1)\chi_1(y) + (A_2+B_2)\chi_2(y) = \int_0^{h_1} G(y,\eta)\,U(\eta)\,d\eta \quad (0 < y < h_1), \tag{2.14}$$

where

$$G(y,\eta) = \sum_{m=1,2}\sum_k k^{-1}\psi_m(y,k)\,\psi_m(\eta,k). \tag{2.15}$$

We emphasize that (2.14) does not hold in $h_1 < y < h_2$ and that the Green's function comprises only the non-propagated modes, corresponding to the positive-real values of k determined by (2.6).

Our problem now can be posed in the following way: supposing the amplitudes of the propagated potentials, A_1+B_1 and A_2+B_2, to be known, determine $U(y)$ to satisfy (2.14) and then determine A_1-B_1 and A_2-B_2 or, equivalently, two linear relations among A_1, A_2, B_1 and B_2, from (2.12).

We generalize the preceding formulation to three dimensions by replacing (2.4) by

$$\phi(x,y;k,l) = \exp\left[\pm(k^2+l^2)^{\frac{1}{2}}x \pm ilz\right]\psi(y,k), \tag{2.16}$$

where l is a positive-real number that may be regarded as determined either by the prescription of an obliquely incident wave (e.g. $l = \kappa\sin\theta$ for a wave travelling at an angle θ with respect to the x-axis) or by appropriate boundary conditions in planes of constant z (e.g. $l = j\pi/b$, $j = 0,1,2,\ldots$, for a channel with walls at $z = 0,b$). The admissible values of k are still determined by (2.6) and (2.8), but we have the additional complication that the mode determined by $k = i\kappa$ is a travelling wave in $x < (>)\,0$ if and only if $l < \kappa_1(\kappa_2)$. This last condition is satisfied automatically for a wave arriving from the deeper water (for which, by definition, $l = \kappa_2\sin\theta < \kappa_2$) by virtue of the inequality $\kappa_1 > \kappa_2$; but a wave arriving from the shallower water suffers total reflexion if $\sin\theta > \kappa_2/\kappa_1$. The situation for a channel of finite width, say $0 < z < b$, is similarly complicated in that only a finite number of the modes can propagate, and this number may be smaller for the deeper channel; e.g. if $\pi/\kappa_1 < b < \pi/\kappa_2$ only the dominant mode ($l = 0$) can propagate in the deeper water, but at least one ($l = \pi/b$) of the higher modes can propagate in the shallower water.

Invoking (2.16), we generalize the two-dimensional formulation to three dimensions by replacing κ_m by $(\kappa_m^2 - l^2)^{\frac{1}{2}}$ in (2.10) and (2.12) and k by $(k^2+l^2)^{\frac{1}{2}}$ in (2.10), (2.13) and (2.15) to obtain

$$-i(\kappa_m^2 - l^2)^{\frac{1}{2}}(A_m - B_m) = \int_0^{h_1} U(\eta)\,\chi_m(\eta)\,d\eta \tag{2.17}$$

and

$$G(y,\eta) = \sum_{m=1,2}\sum_k (k^2+l^2)^{-\frac{1}{2}}\psi_m(y,k)\,\psi_m(\eta,k). \tag{2.18}$$

If $l > \kappa_2$, we must take $A_2 = B_2 = 0$ and insert the additional term

$$(-\kappa_2^2+l^2)^{-\frac{1}{2}}\psi_2(y, i\kappa_2)\,\psi_2(\eta, i\kappa_2)$$

in the summation of (2.18).

3. The scattering matrix

We now introduce the linear matrices

$$\mathbf{A} = \{A_1, A_2\}, \quad \mathbf{B} = \{B_1, B_2\}, \tag{3.1}$$

$$\boldsymbol{\chi}(y) = \{\chi_1(y), \chi_2(y)\}, \tag{3.2}$$

and the diagonal matrix

$$\boldsymbol{\kappa}(l) = [\delta_{mn}(\kappa_m^2 - l^2)^{\frac{1}{2}}], \tag{3.3}$$

and rewrite (2.17) and (2.14) as

$$-i\boldsymbol{\kappa}(l)(\mathbf{A}-\mathbf{B}) = \int_0^{h_1} \boldsymbol{\chi}(\eta)\, U(\eta)\, d\eta \tag{3.4}$$

and
$$(\mathbf{A}+\mathbf{B})\boldsymbol{\chi}(y) = \int_0^{h_1} G(y,\eta)\, U(\eta)\, d\eta \quad (0 < y < h_1). \tag{3.5}$$

We infer from (3.4) and (3.5) that $\mathbf{A}-\mathbf{B}$ and $\mathbf{A}+\mathbf{B}$ must be linearly related and define the scattering matrix $\mathbf{S}$ such that

$$\boldsymbol{\kappa}(\mathbf{A}-\mathbf{B}) = i\mathbf{S}(\mathbf{A}+\mathbf{B}), \quad \mathbf{S} = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix}, \tag{3.6}$$

which implies

$$\mathbf{B} = \mathbf{TA}, \quad \mathbf{T} = (\boldsymbol{\kappa}+i\mathbf{S})^{-1}(\boldsymbol{\kappa}-i\mathbf{S}). \tag{3.7}$$

Similarly, the unknown velocity $U(y)$ must be linear in $\mathbf{A}+\mathbf{B}$, regarded as a description of the excitation; accordingly, we define the normalized velocity $\mathbf{u}$ such that

$$U(y) = (\mathbf{A}+\mathbf{B})\,\mathbf{u}(y), \quad \mathbf{u}(y) = \{u_1(y),\, u_2(y)\}. \tag{3.8}$$

Substituting (3.8) into (3.5) and invoking the linear independence of A_1+B_1 and A_2+B_2, we obtain the uncoupled integral equations

$$\chi_m(y) = \int_0^{h_1} G(y,\eta)\, u_m(\eta)\, d\eta \quad (0 < y < h_1) \tag{3.9}$$

for the determination of $\mathbf{u}(y)$. Substituting (3.8) into (3.4) and invoking (3.6), we obtain

$$S_{mn} = \int_0^{h_1} \chi_m(\eta)\, u_n(\eta)\, d\eta \quad (m, n = 1, 2). \tag{3.10}$$

We now have reduced our problem to the solution of the integral equations (3.9) and the subsequent determination of the scattering matrix from (3.10). We remark that $\mathbf{u}$ and $\mathbf{S}$ are real, even though $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{T}$ are generally complex. We could transform (3.4) and (3.5) to an infinite set of linear algebraic equations with complex coefficients by expanding $U(y)$ in the $\psi_1(y, k)$, which form a complete set of orthonormal functions in the interval $y = (0, h_1)$; similarly, we could transform (3.9) to two, uncoupled, infinite sets of linear algebraic equations with real coefficients by expanding both $u_1(y)$ and $u_2(y)$ in the $\psi_1(y, k)$. The former procedure was adopted by Newman (1965*a*). We rest content with the more direct construction of approximate solutions.

The complex reflexion and transmission coefficients for the step are comprised by $\mathbf{T}$ after appropriate normalization (in which respect, we recall the factor of $\operatorname{sgn} x$ in (2.10)). For example, the prescription $\mathbf{A} = \{0, 1\}$ implies the reflexion coefficient B_2 and the (velocity-potential) transmission coefficient $-B_1$, and conversely for $\mathbf{A} = \{1, 0\}$. The free-surface displacements are given by (for normal incidence)

$$\eta_m(x,t) = \operatorname{Re}\left[(-i\sigma/g)\,\phi_m(x,0)\,e^{-i\sigma t}\right] \tag{3.11a}$$

$$\sim (\sigma/g)\,\chi_m(0)\operatorname{sgn} x \operatorname{Im}\left[A_m e^{-i(\sigma t+\kappa_m|x|)} + B_m e^{-i(\sigma t-\kappa_m|x|)}\right], \tag{3.11b}$$

so that the transmission coefficient for the free-surface displacement is $-B_1\chi_1(0)/\chi_2(0)$. This last factor applies also for oblique incidence, provided that $l < \kappa_2$.

4. Plane-wave approximation

Lamb (1932, §176) obtained long-wave approximations to the reflexion and transmission coefficients for a step by assuming (*a*) plane waves, and (*b*) shallow-water theory, after which considerations of continuity yield the required relations. [Bartholomeusz (1958) has shown that (*a*) and (*b*) are consistent in the sense that (*b*) implies (*a*) as $Kh_{1,2} \to 0$.] Assumption (*a*) implies the neglect of the non-propagated modes, $\psi_m(y,k)$ for k real, or, equivalently, $G = 0$ in the preceding formulation. We generalize Lamb's results by invoking this approximation for unrestricted $Kh_{1,2}$.

Referring to the discussion following (3.10), we infer that the neglect of $\psi_1(y,k)$ for all real k implies the truncated expansion

$$U(y) = C\chi_1(y). \tag{4.1}$$

Substituting (4.1) into (3.4), we place the result in the form

$$\boldsymbol{\kappa}(\mathbf{A}-\mathbf{B}) = iC\{1, \lambda N\}, \tag{4.2}$$

where

$$\lambda N = \int_0^{h_1} \chi_1(y)\,\chi_2(y)\,dy \tag{4.3a}$$

$$= 2K\kappa_2(\kappa_1^2-\kappa_2^2)^{-1}(Kh_1+\sinh^2\kappa_1 h_1)^{-\frac{1}{2}} \times (Kh_2+\sinh^2\kappa_2 h_2)^{-\frac{1}{2}}\sinh\left[\kappa_2(h_2-h_1)\right]. \tag{4.3b}$$

Anticipating the simplification of $\mathbf{T}$, we choose

$$\lambda = \left[(\kappa_2^2-l^2)/(\kappa_1^2-l^2)\right]^{\frac{1}{4}}. \tag{4.4}$$

Setting $G = 0$ in (3.5), we obtain

$$(\mathbf{A}+\mathbf{B})\boldsymbol{\chi}(y) = 0 \quad (0 < y < h_1). \tag{4.5}$$

Only the first moment of (4.5), namely (the result of multiplying (4.5) through by $\chi_1(y)$ and integrating over $y = (0, h_1)$)

$$\{1, \lambda N\}(\mathbf{A}+\mathbf{B}) = 0 \tag{4.6a}$$

is significant within the present approximation. Eliminating C between the two components of (4.2), we obtain the complementary moment

$$\{-\lambda N, 1\}\boldsymbol{\kappa}(\mathbf{A}-\mathbf{B}) = 0. \tag{4.6b}$$

It is evident that (4.6a, b) do not permit the construction of a non-singular relation of the type (3.6). Nevertheless, we can construct a relation of the type (3.7), namely

$$\mathbf{B} = \mathbf{TA}, \quad \mathbf{T} = \left(\frac{1}{1+N^2}\right)\begin{bmatrix} N^2-1 & -2\lambda N \\ -2\lambda^{-1}N & 1-N^2 \end{bmatrix}. \tag{4.7}$$

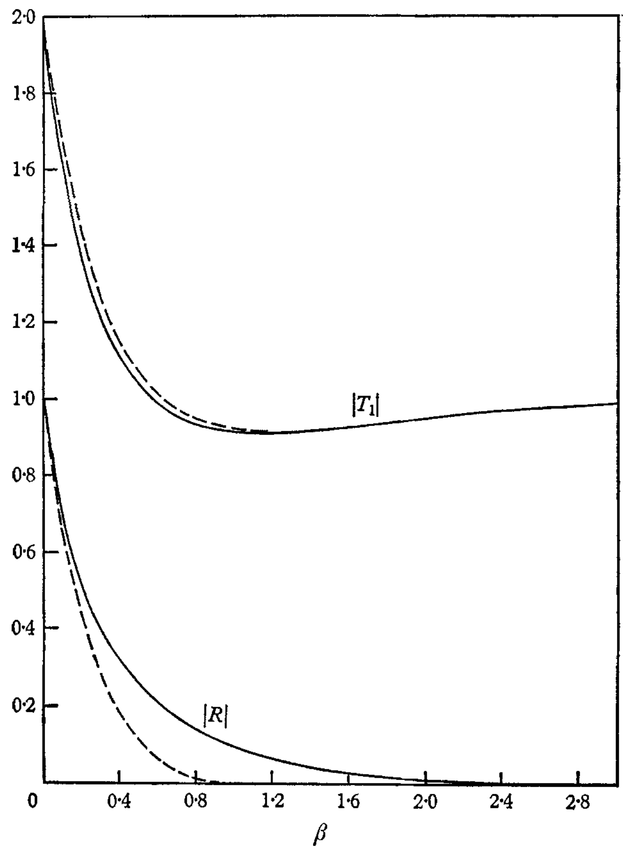

FIGURE 2. The plane-wave (dashed) and variational approximations (solid) to the reflexion and transmission coefficients for an infinite step, as determined from (4.8), (6.12) and (6.13).

We note the special cases

$$\mathbf{A} = \{0, 1\} \Rightarrow \mathbf{B} = \left\{\frac{-2\lambda N}{1+N^2}, \quad \frac{1-N^2}{1+N^2}\right\} \tag{4.8}$$

and

$$\mathbf{A} = \{1, 0\} \Rightarrow \mathbf{B} = \left\{\frac{N^2-1}{N^2+1}, \quad \frac{-2\lambda^{-1}N}{N^2+1}\right\}, \tag{4.9}$$

corresponding to incident waves from the right and from the left, respectively. We identify N^2 as an appropriately normalized impedance ratio for the two domains.

The reflexion and transmission coefficients given by (4.8) for an infinite step are compared with the more accurate approximations of §6 (which, in turn, are in agreement with Newman's (1965a) results) in figure 2. The approximation to the transmission coefficient is within 5 % of Newman's result for all wavelengths, but the corresponding approximation to the reflexion coefficient is satisfactory only for small values of $\beta \equiv \kappa_1 h_1$ (using the expansions developed in §6, we find that the approximations of (4.8) give the correct coefficients of β^0 and β as $\beta \to 0$). We emphasize that the plane-wave approximation cannot give the phase shifts of the transmitted and reflected waves.

5. Variational formulation

We now construct variational integrals for the S_{mn}. Multiplying (3.9) by $u_n(y)$, integrating over $y = (0, h_1)$, and comparing the result to (3.10), we obtain

$$S_{mn} = \int_0^{h_1} \int_0^{h} u_m(y)\, G(y,\eta)\, u_n(\eta)\, dy\, d\eta. \tag{5.1}$$

Combining (5.1) and (3.10), we obtain

$$\frac{1}{S_{mn}} = \frac{\displaystyle\int_0^{h_1} \int_0^{h_1} u_m(y)\, G(y,\eta)\, u_n(\eta)\, dy\, d\eta}{\displaystyle\int_0^{h_1} \chi_m(y)\, u_n(y)\, dy \int_0^{h_1} \chi_n(\eta)\, u_m(\eta)\, d\eta}. \tag{5.2}$$

We remark that (5.2) (a) is invariant under a scale transformation of $\mathbf{u}(y)$, (b) is stationary with respect to first-order variations of $\mathbf{u}$ about the true solution to the integral equation (3.9), and (c) implies the reciprocity relation

$$S_{21} = S_{12} \tag{5.3}$$

by virtue of the symmetry of $G(y, \eta)$.

We could expand $\mathbf{u}$ in the $\psi_1(y, k)$ and then determine the coefficients in the expansion by a systematic invocation of the variational principle (b) above. The resulting algebraic equations would be equivalent to those obtained through the direct solution of (3.9), as suggested in the discussion following (3.10). The power of the variational principle, however, lies primarily in direct approximations. The simplest rational approximation appears to be that of (4.1) or, equivalently,

$$u_m(y) = C_m \chi_1(y) \quad (m = 1, 2). \tag{5.4}$$

Substituting (5.4) into (5.2) and invoking the orthogonality of $\chi_1(y)$ with respect to the remaining $\psi_1(y, k)$, we place the results in the form

$$S_{11} = S_{12}/\lambda N = S_{22}/(\lambda N)^2 \equiv (\kappa_1^2 - l^2)^{\frac{1}{2}}/X, \tag{5.5}$$

and
$$X = (\kappa_1^2 - l^2)^{\frac{1}{2}} \int_0^{h_1} \int_0^{h_1} \chi_1(y)\, G(y,\eta)\, \chi_1(\eta)\, dy\, d\eta \tag{5.6a}$$

$$= (\kappa_1^2 - l^2)^{\frac{1}{2}} \sum_k (k^2 + l^2)^{-\frac{1}{2}} \left[\int_0^{h_1} \chi_1(y)\, \psi_2(y, k)\, dy\right]^2 \tag{5.6b}$$

$$= 4K^2 (\kappa_1^2 - l^2)^{\frac{1}{2}} (Kh_1 + \sinh^2 \kappa_1 h_1)^{-1} \sum_k k^2 (k^2 + l^2)^{-\frac{1}{2}} (\kappa_1^2 + k^2)^{-2} \times (Kh_2 - \sin^2 kh_2)^{-1} \sin^2 [k(h_2 - h_1)], \tag{5.6c}$$

where λN is given by (4.3), and the summation is over the positive-real values of k obtained by setting $h = h_2$ in (2.6).

We remark that (5.5) implies

$$|S_{mn}| = S_{11}S_{22} - S_{12}^2 \equiv 0, \tag{5.7}$$

in consequence of which (3.6) is singular, just as in the plane-wave approximation of the preceding section (indeed, (5.4) *et seq.* represent a variational improvement of the plane-wave approximation). It remains true that (3.7) is non-singular. Substituting (5.5) into (3.7), we obtain

$$\mathbf{T} = \left(\frac{1}{1+N^2-iX}\right)\begin{bmatrix} N^2-1-iX & -2\lambda N \\ -2\lambda^{-1}N & 1-N^2-iX \end{bmatrix}. \tag{5.8}$$

We observe that all effects of the higher modes on $\mathbf{T}$ are incorporated in the parameter X and that (5.8) reduces to the plane-wave approximation (4.7) for $X = 0$.

6. Infinite step

We consider further the special case $h_2 = \infty$ in order to compare our variational approximation with Newman's (1965*a*) more accurate results. Recalling that $h_2 = \infty$ implies $\kappa_2 = K$, we introduce the dimensionless parameters

$$\alpha = Kh_1, \quad \beta = \kappa_1 h_1, \quad \gamma = lh_1 \equiv \alpha \sin\theta \equiv \beta\sin\phi, \tag{6.1}$$

and rewrite (2.8) and (4.4) in the forms

$$\alpha = \beta\tanh\beta \tag{6.2a}$$

$$= \beta^2[1-\tfrac{1}{3}\beta^2+O(\beta^4)] \quad (\beta\to 0) \tag{6.2b}$$

$$\sim \beta[1-2e^{-2\beta}+O(e^{-4\beta})] \quad (\beta\to\infty), \tag{6.2c}$$

and

$$\lambda = [(\alpha^2-\gamma^2)/(\beta^2-\gamma^2)]^{\frac{1}{2}} \tag{6.3a}$$

$$= (\tanh\beta)^{\frac{1}{2}}(1+\operatorname{sech}^2\beta\tan^2\theta)^{-\frac{1}{2}}. \tag{6.3b}$$

We also find it expedient to regard N and X as functions of β and θ with α and γ (or ϕ) given by (6.1).

Substituting (2.7) and (2.9*a*, *b*) into (4.3*a*) and (5.6*b*) and replacing the summation in (5.6*b*) by integration over $u = ky = (0,\infty)$, we obtain

$$N(\beta,\theta) = N(\beta,0)\,(1+\operatorname{sech}^2\beta\tan^2\theta)^{\frac{1}{2}}, \tag{6.4}$$

$$N(\beta,0) = e^{-\alpha}(\beta+\tfrac{1}{2}\sinh 2\beta)^{-\frac{1}{2}}\sinh 2\beta \tag{6.5a}$$

$$= (2\beta)^{\frac{1}{2}}[1-\tfrac{1}{2}\beta^2+O(\beta^4)] \tag{6.5b}$$

$$\sim 1+O(\beta^2 e^{-4\beta}), \tag{6.5c}$$

and

$$X(\beta,\theta) = \frac{4\beta^2\cos\phi}{\pi(\beta+\frac{1}{2}\sinh 2\beta)}\int_0^\infty \frac{(\alpha\cos u+u\sin u)^2u^2\,du}{(\alpha^2+u^2)(\beta^2+u^2)^2(\gamma^2+u^2)^{\frac{1}{2}}}. \tag{6.6}$$

It does not appear possible to express the integral of (6.6) in finite terms of tabulated functions if $\gamma \neq 0$. For $\gamma = 0$, we expand $(\alpha^2+u^2)^{-1}(\beta^2+u^2)^{-2}$ in partial fractions to obtain

$$X(\beta,0) = \pi^{-1}\{\operatorname{Ei}(2\beta)-\operatorname{Ei}(-2\beta)-N^2[\operatorname{Ei}(2\alpha)+\tfrac{1}{2}e^{2\alpha}\operatorname{sech}^2\beta]\} \tag{6.7a}$$

$$= (2/\pi)\beta[L-\beta^2(L-\tfrac{5}{18})+O(\beta^4 L)] \tag{6.7b}$$

$$\sim (4/\pi)e^{-2\beta}[1+O(1/\beta^2)], \tag{6.7c}$$

where
$$\mathrm{Ei}(u) = \int_{-\infty}^{u} v^{-1} e^{v} dv \tag{6.8}$$
and
$$L = 0{\cdot}230 + 2\log(1/\beta). \tag{6.9}$$

The results (6.2*a*), (6.4*a*) and (6.7*a*), for α, N and X, are plotted in figure 3.

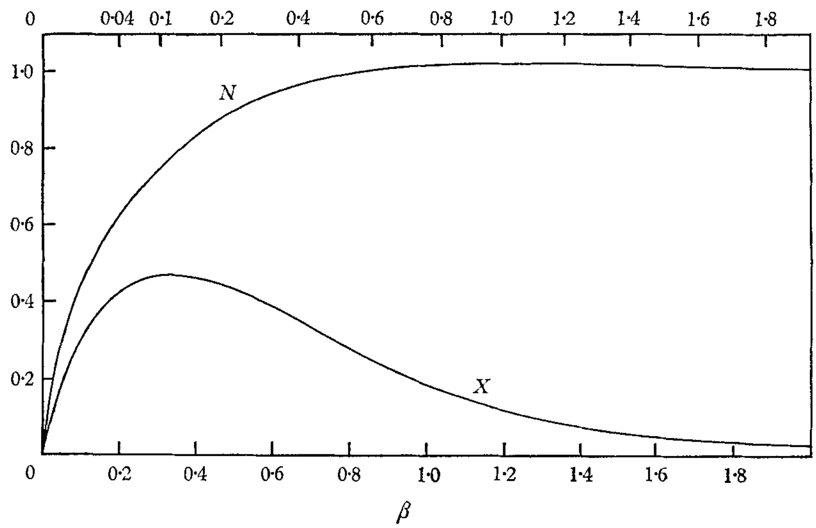

FIGURE 3. The results for N and X, as determined from (6.2), (6.5) and (6.7).

For $\gamma \neq 0$, we consider only the limiting cases $\alpha \to 0$ and $\alpha \to \infty$. Considering first $\alpha \to \beta^2 \to 0$, we separate (6.6) into two integrals over $u = (0, \beta)$, where we approximate $\alpha \cos u + u \sin u$ by $\alpha + u^2$, and $u = (\beta, \infty)$, where we approximate $(\gamma^2 + u^2)^{\frac{1}{2}}$ by u, to obtain

$$X(\beta, \theta) = (2\beta/\pi)\left[L + \log(2\operatorname{cosec}\theta) - \sec\theta \log(\cot\tfrac{1}{2}\theta)\right]\left[1 + O(\beta^2)\right]. \tag{6.10}$$

For $\alpha \to \beta \to \infty$, we proceed according to

$$X(\beta, \theta) \sim \frac{16\beta^2 e^{-2\beta}\cos\phi}{\pi}\int_0^\infty \frac{(\beta\cos u + u \sin u)^2 u^2 du}{(\beta^2 + u^2)^3 (u^2 + \beta^2 \sin^2\phi)^{\frac{1}{2}}}\left[1 + O(\beta^2 e^{-2\beta})\right] \tag{6.11a}$$

$$= \frac{8 e^{-2\beta}\cos\theta}{\pi}\left[\int_0^\infty \frac{v^2 dv}{(1 + v^2)^2 (v^2 + \sin^2\theta)^{\frac{1}{2}}} + O\left(\frac{1}{\beta^2}\right)\right] \tag{6.11b}$$

$$= (4/\pi) e^{-2\beta}\left[\sec\theta - \tan^2\theta \log(\cot\tfrac{1}{2}\theta) + O(1/\beta^2)\right], \tag{6.11c}$$

where (6.11*b*) follows from (6.11*a*) after the change of variable $u = \beta v$ and the neglect of the oscillating portion of the integrand. We observe that the bracketed term in (6.10) decreases monotonically from

$$2\log(1/\beta) + 0{\cdot}230 \quad \text{to} \quad 2\log(1/\beta) - 0{\cdot}077$$

as θ increases from 0 to $\frac{1}{2}\pi$, whereas the bracketed term in (6.11*c*) decreases monotonically from 1 to 0.

We test the approximation of § 5 by calculating the reflexion and transmission coefficients for a surface wave normally incident from deep water. Calculating

$\mathbf{B} = T\{0, 1\}$ with the aid of (5.8) and renormalizing the transmission coefficient in accordance with the last paragraph of §3, we obtain

$$R_1 = (1-N^2-iX)/(1+N^2-iX) \tag{6.12a}$$

$$= 1-4\beta+8\beta^2+4\beta^3[(4/\pi^2)L^2-3]-(8i/\pi)\beta^2(1-4\beta)L+O(\beta^4L) \tag{6.12b}$$

$$\sim -(2i/\pi)e^{-2\alpha}+O(\alpha^2e^{-4\alpha}) \tag{6.12c}$$

and

$$T_1 = 2\lambda N(\alpha+\sinh^2\beta)^{-\frac{1}{2}}(1+N^2-iX)^{-1}\cosh\beta \tag{6.13a}$$

$$= 2-4\beta+\beta^2[\tfrac{23}{3}-(8/\pi^2)L^2]+(4i\beta/\pi)(1-4\beta)L+O(\beta^3L) \tag{6.13b}$$

$$\sim 1+[1-2\alpha+(2i/\pi)]e^{-2\alpha}+O(\alpha^2e^{-4\alpha}), \tag{6.13c}$$

where $\alpha = \beta$ within the approximations of (6.12c) and (6.13c). We observe that $R_1 = |R|\,e^{i\delta R_1}$ and $T_1 = |T_1|\,e^{i\delta T}$ in Newman's (1965a) paper and that (6.12b) and (6.13b) are in agreement with, although carried to higher order than, Newman's (6.5) and (6.6) after correction of what appears to be a typographical error (π should be replaced by π^2 in his (6.5)).

The magnitudes $|R|$ and $|T_1|$, as determined by (6.12a) and (6.13a), are plotted in figure 2. These magnitudes, and also the corresponding phase angles δR_1 and δT_1, agree with Newman's results within the accuracy with which his graphs can be read.

We remark that (6.12) and (6.13) reduce to the plane-wave approximations implied by (4.8) if we set $X = 0$ in (6.12a) and (6.13a), $L = 0$ in (6.12b) and (6.13b), and $i = 0$ in (6.12c) and (6.13c).

The generalizations of (6.12) and (6.13) for a wave obliquely incident from deep water are

$$R_1 = 1-4\beta\sec\theta+8\beta^2\sec^2\theta-4i\beta X\sec\theta+O(\beta^3) \tag{6.14a}$$

$$\sim -e^{-2\alpha}\{\tan^2\theta+(2i/\pi)[\sec\theta-\tan^2\theta\log(\cot\tfrac{1}{2}\theta)]+O(1/\alpha^2)\} \tag{6.14b}$$

and

$$T_1 = 2-4\beta\sec\theta+2iX+O(\beta^2) \tag{6.15a}$$

$$\sim 1+\{1-2\alpha+(2i/\pi)[\sec\theta-\tan^2\theta\log(\cot\tfrac{1}{2}\theta)]+O(1/\alpha^2)\}e^{-2\alpha}, \tag{6.15b}$$

where X in (6.14a) and (6.15a) is given by (6.10). The θ dependence of X is less important than that of N for intermediate values of β (see remarks following (6.11)), which suggests that good approximations to R_1 and T_1 for oblique incidence and intermediate values of β can be obtained simply by multiplying N^2 by $(1+\operatorname{sech}^2\beta\tan^2\theta)^{\frac{1}{2}}$ in (6.12a) and (6.13a) [cf. (6.4); λN is independent of θ].

Summing up, the variational approximations of (5.5) and (6.6) give at least the first three terms in the expansions of Newman's results about $\alpha = 0$, give the same limiting values as $\alpha \to \infty$, and are in close numerical agreement for all wavelengths. Recalling that Newman's calculations involved the numerical solution of between 10 and 80 simultaneous equations, we conclude that the variational approximation given by (5.4)–(5.6) is both powerful and economical.

7. End correction for shelf

We illustrate the flexibility of the scattering-matrix formulation by supposing that the shallower water in $x < 0$ terminates at a vertical cliff, say $x = -d$, and calculating the effective length that must be added to d to account for the phase shift associated with the discontinuity at $x = 0$.

Invoking the boundary condition

$$\phi_x = 0 \quad (x = -d), \tag{7.1}$$

we infer from the three-dimensional generalization of (2.10) that A_1 and B_1 must be of the form

$$A_1 = \tfrac{1}{2}C_1 e^{i\delta}, \quad B_1 = \tfrac{1}{2}C_1 e^{-i\delta}, \tag{7.2}$$

where C_1 is real, and

$$\delta = (\kappa_1^2 - l^2)^{\frac{1}{2}} d. \tag{7.3}$$

Choosing $A_2 = 1$, we also define

$$R = B_2, \quad T = -C_1\chi_1(0)/\chi_2(0). \tag{7.4}$$

Substituting (7.2) and (7.4) into (3.6) and invoking the approximation (5.5), we obtain

$$T = \frac{2\lambda N[\chi_1(0)/\chi_2(0)]}{\cos\delta - X\sin\delta - iN^2\sin\delta} \equiv |T|\, e^{i\tau} \tag{7.5}$$

and

$$R = e^{2i\tau}, \tag{7.6}$$

where

$$\tau = \cot^{-1}[(X - \cot\delta)/N^2] \tag{7.7}$$

is the phase angle of T.

We define resonance as that condition for which $\tau = \frac{1}{2}\pi$ (or, more generally, $\tau = (n+\frac{1}{2})\pi$), which implies

$$\delta + \tan^{-1} X = \tfrac{1}{2}\pi + n\pi \quad (n = 0, 1, 2, \ldots), \tag{7.8}$$

from which we infer that the effect of the non-planar modes is equivalent to an incremental shelf length of

$$d_1 = (\kappa_1^2 - l^2)^{-\frac{1}{2}} \tan^{-1} X. \tag{7.9}$$

We note that the extrema of $|T|$ correspond to

$$\delta + \tfrac{1}{2}\tan^{-1}\left(\frac{2X}{1 - N^4 - X^2}\right) = \tfrac{1}{2}\pi + n\pi \quad (n = 0, 1, 2, \ldots), \tag{7.10}$$

rather than to (7.8).

The most interesting case is $h_2 = \infty$ and $\beta \to 0$, for which (7.5) reduces to

$$T = 2[\cos\delta - (X + 2i\beta\sec\theta)\sin\delta]^{-1}[1 + O(\beta^2)], \tag{7.11}$$

where X is given by (6.10) and is $O(\beta)$. The criteria of (7.8) and (7.10) now are equivalent within $1 + O(\beta^2)$ and imply

$$|T|_{\max} = \beta^{-1}\cos\theta \quad \text{at} \quad \delta + X = \tfrac{1}{2}\pi + n\pi, \tag{7.12}$$

and

$$d_1/h_1 = X/\beta = (2/\pi)\,[0{\cdot}230 + \log(2/\beta^2\sin\theta) - \sec\theta\log(\cot\tfrac{1}{2}\theta) + O(\beta^2)]. \tag{7.13}$$

[Dr Newman has pointed out that the results of this last section are related to his own results for a 'long symmetrical obstacle' (Newman 1965*b*). Indeed, symmetry implies that our problem is identical with that of a rectangular step of length $2d$ subjected to identical incident waves from $x = \pm\infty$.]

The ratio d_1/h_1 for $h_2 = \infty$ and $\theta = 0$ is plotted in figure 4. We emphasize that, although d_1/h_1 is logarithmically infinite, $d_1 \to 0$ in the limit $\beta \to 0$; accordingly, the effect of trapped modes on the resonant width of a shallow shelf may be negligible in many applications.

This research was partially supported by the Office of Naval Research under Contract Nonr 2216(29), and by the National Science Foundation, under Contract NSF-GP-2414.

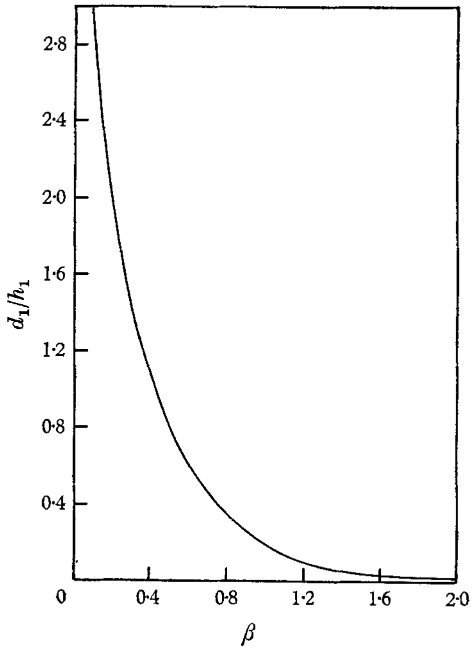

FIGURE 4. The incremental shelf length d_1 for $h_2 = \infty$ and $\theta = 0$; d_1 is equivalent, in a plane-wave calculation of resonance, to the effects of the non-planar modes.

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